XXII. *A Fourth Memoir upon Quantics.* By ARTHUR CAYLEY, *Esq.*, *F.R.S.*

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THE object of the present memoir is the further development of the theory of binary quantics; it should therefore have preceded so much of my third memoir, t. 147 (1857), p. 627, as relates to ternary quadrics and cubics. The paragraphs are numbered continuously with those of the former memoirs. The first three paragraphs, Nos. 62 to 64, relate to quantics of the general form $(*\between x, y, ..)^m$, and they are intended to complete the series of definitions and explanations given in Nos. 54 to 61 of my third memoir; Nos. 68 to 71, although introduced in reference to binary quantics, relate or may be considered as relating to quantics of the like general form. But with these exceptions the memoir relates to binary quantics of any order whatever: viz. No. 65 to 80 relate to the covariants and invariants of the degrees 2, 3 and 4; Nos. 81 and 82 (which are introduced somewhat parenthetically) contain the explanation of a process for the calculation of the invariant called the Discriminant; Nos. 83 to 85 contain the definitions of the Catalecticant, the Lambdaic and the Canonisant, which are functions occurring in Professor SYLVESTER'S theory of the reduction of a binary quantic to its canonical form; and Nos. 86 to 91 contain the definitions of certain covariants or other derivatives connected with BEZOUT'S abbreviated method of elimination, due for the most part to Professor SYLVESTER, and which are called Bezoutiants, Cobezoutiants, &c. I have not in the present memoir in any wise considered the theories to which the catalecticant, &c. and the last-mentioned other covariants and derivatives relate; the design is to point out and precisely define the different covariants or other derivatives which have hitherto presented themselves in theories relating to binary quantics, and so to complete, as far as may be, the explanation of the terminology of this part of the subject.

62. If we consider a quantic

$$(a, b, ..\between x, y, ...)^m$$

and an adjoint linear form, the operative quantic

$$(a, b, ..\between \partial_\xi, \partial_\eta, ..)^m,$$

or more generally the operative quantic obtained by replacing in any covariant of the given quantic the facients $(x, y, ..)$ by the symbols of differentiation $(\partial_\xi, \partial_\eta, ...)$ (which operative quantic is, so to speak, a contravariant operator), may be termed the *Provector*; and the Provector operating upon any contravariant gives rise to a contravariant, which may of course be an invariant. Any such contravariant, or rather such contravariant considered as so generated, may be termed a *Provectant*; and in like manner the operative quantic obtained by replacing in any contravariant of the given quantic the facients (ξ, η..) by the symbols of differentiation (∂_x, ∂_y, ...) (which operative quantic is a covariant operator), is termed the *Contraprovector*; and the contraprovector operating upon any covariant gives rise to a covariant, which may of course be an invariant. Any such covariant, or rather such covariant considered as so generated, may be termed a *Contraprovectant*.

In the case of a binary quantic,

$$(a, b, .. \between x, y)^m,$$

the two theorems coalesce together, and we may say that the operative quantic

$$(a, b, .. \between \partial_y, -\partial_x)^m,$$

or more generally the operative quantic obtained by replacing in any covariant of the given quantic the facients (x, y) by the symbols of differentiation (∂_y, $-\partial_x$) (which is in this case a covariant operator), may be termed the Provector. And the Provector operating on any covariant gives a covariant (which as before may be an invariant), and which considered as so generated may be termed the Provectant.

63. But there is another allied theory. If in the quantic itself or in any covariant we replace the facients (x, y, ...) by the first derived functions (∂_ξP, ∂_ηP...) of any contravariant P of the quantic, we have a new function which will be a contravariant of the quantic. In particular, if in the quantic itself we replace the facients (x, y...) by the first derived functions (∂_ξP, ∂_ηP, ..) of the Reciprocant, then the result will contain as a factor the Reciprocant, and the other factor will be also a contravariant. And similarly, if in any contravariant we replace the facients (ξ, η, ..) by the first derived functions (∂_xW, ∂_yW, ..) of any covariant W (which may be the quantic itself) of the quantic U, we have a new function which will be a covariant of the quantic. And in particular if in the Reciprocant we replace the facients (ξ, η, ...) by the first derived functions (∂_xU, ∂_yU..) of the quantic, the result will contain U as a factor, and the other factor will be also a covariant. In the case of a binary quantic $(a, b, .. \between x, y)^m$ the two theorems coalesce and we have the following theorem, viz. if in the quantic U or any covariant the facients (x, y) are replaced by the first derived functions (∂_yW, $-\partial_x$W) of a covariant W, the result will be a covariant; and if in the quantic U the facients (x, y) are replaced by the first derived functions (∂_yU, $-\partial_x$U) of the quantic, the result will contain U as a factor, and the other factor will be also a covariant.

Without defining more precisely, we may say that the function obtained by replacing as above the facients of a covariant or contravariant by the first derived functions of a contravariant or covariant is a *Transmutant* of the first-mentioned covariant or contravariant.

64. Imagine any two quantics of the same order, for instance, the two quantics

$$U=(a, b, ... \between x, y..)^m,$$
$$V=(a', b', ... \between x, y..)^m,$$

then any quantic such as $\lambda U+\mu V$ may be termed an *Intermediate* of the two quantics;

and a covariant of $\lambda U+\mu V$, if in such covariant we treat λ, μ as facients, will be a quantic of the form

$$(A, B, .. B', A' \between \lambda, \mu)^n,$$

where the coefficients (A, B, .. B', A') will be covariants of the quantics U, V, viz. A will be a covariant of the quantic U alone; A' will be the same covariant of the quantic V alone, and the other coefficients (which in reference to A, A' may be termed the *Connectives*) will be covariants of the two quantics; and any coefficient may be obtained from the one which precedes it by operating on such preceding coefficient with the combinantive operator

$$a'\partial_a+b'\partial_b+\ldots,$$

or from the one which succeeds it by operating on such succeeding coefficient with the combinantive operator

$$a\partial_{a'}+b\partial_{b'}+\ldots$$

the result being divided by a numerical coefficient which is greater by unity than the index of μ or (as the case may be) λ in the term corresponding to the coefficient operated upon. It may be added, that any invariant in regard to the facients (λ, μ) of the quantic

$$(A, B, ..B', A' \between \lambda, \mu)^n$$

is not only a covariant, but it is also a combinant of the two quantics U, V.

As an example, suppose the quantics U, V are the quadrics

$$(a, b, c \between x, y)^2 \text{ and } (a', b', c' \between x, y)^2,$$

then the quadrinvariant of

$$\lambda U+\mu V \text{ is } (\lambda a+\mu a')(\lambda c+\mu c')-(\lambda b+\mu b')^2,$$

which is equal to

$$(ac-b^2,\ ac'-2bb'+ca',\ a'c'-b'^2 \between \lambda, \mu)^2,$$

and $ac'-2bb'+ca'$ is the connective of the two discriminants $ac-b^2$ and $a'c'-b'^2$.

65. The law of reciprocity for the number of the invariants of a binary quantic*, leads at once to the theorems in regard to the number of the quadrinvariants, cubinvariants and quartinvariants of a binary quantic of a given degree, first obtained by the method in the second part of my original memoir†. Thus a quadric has only a single invariant, which is of the degree 2; hence, by the law of reciprocity, the number of quadrinvariants of a quantic of the order m is equal to the number of ways in which m can be made up with the part 2, which is of course unity or zero, according as m is even or odd. And we conclude that

The quadrinvariant exists only for quantics of an even order, and for each such quantic there is one, and only one, quadrinvariant.

66. Again, a cubic has only one invariant, which is of the degree 4, and the number of cubinvariants of a quantic of the degree m is equal to the number of ways in which m can be made up with the part 4. Hence

* Introd. Memoir, No. 20.

† Ibid. Nos. 10–17.

A cubinvariant only exists for quantics of an evenly even order, and for each such quantic there is one, and only one, cubinvariant.

67. But a quartic has two invariants, which are of the degrees 2 and 3 respectively, and the number of quartinvariants of a quantic of the degree m is equal to the number of ways in which m can be made up with the parts 2 and 3. When m is even, there is of course a quartinvariant which is the square of the quadrinvariant, and which, if we attend only to the irreducible quartinvariants, must be excluded from consideration. The preceding number must therefore, when m is even, be diminished by unity. The result is easily found to be

Quartinvariants exist for a quantic of any order, even or odd, whatever, the quadric and the quartic alone excepted; and according as the order of the quantic is

$$6g,\quad 6g+1,\quad 6g+2,\quad 6g+3,\quad 6g+4,\quad 6g+5,$$

the number of quartinvariants is

$$g\,,\quad g\,,\quad g\,,\quad g+1\,,\quad g\,,\quad g+1\,.$$

In particular, for the orders

$$2,\ 3,\ 4,\ 5;\ 6,\ 7,\ 8,\ 9,\ 10,\ 11;\ 12, \&c.,$$

the numbers are

$$0,\ 1,\ 0,\ 1;\ 1,\ 1,\ 1,\ 2,\ 1,\ 2;\ 2, \&c.$$

So that the nonic is the lowest quantic which has more than one quartinvariant.

68. But the whole theory of the invariants or covariants of the degrees 2, 3, 4 is most easily treated by the method above alluded to, contained in the second part of my original memoir; and indeed the method appears to be the appropriate one for the treatment of the theory of the invariants or covariants of any given degree whatever, although the application of it becomes difficult when the degree exceeds 4. I remark, in regard to this method, that it leads naturally, and in the first instance, to a special class of the covariants of a system of quantics, viz. these covariants are linear functions of the derived functions of any quantic of the system. (It is hardly necessary to remark that the derived functions referred to are the derived functions of any order of the quantic with regard to the facients.) Such covariants may be termed *tantipartite* covariants; but when there are only two quantics, I use in general the term *lineo-linear*. The tantipartite covariants, while the system remains general, are a special class of covariants, but by particularizing the system we obtain all the covariants of the particularized system. The ordinary case is when all the quantics of the system reduce themselves to one and the same quantic, and the method then gives all the covariants of such single quantic. And while the order of the quantic remains indefinite, the method gives covariants (not invariants); but by particularizing the order of the quantic in such manner that the derived functions become simply the coefficients of the quantic, the covariants become invariants: the like applies of course to a system of two or more quantics.

69. To take the simplest example, in seeking for the covariants of a single quantic U, we in fact have to consider two quantics U, V. An expression such as $\overline{12}\,UV$ is a lineo-linear covariant of the two quantics; its developed expression is

$$\partial_x U . \partial_y V - \partial_y U . \partial_x V,$$

which is the Jacobian. In the particular case of two linear functions $(a, b \between x, y)$ and $(a', b' \between x, y)$, the lineo-linear covariant becomes the lineo-linear invariant $ab'-a'b$, which is the Jacobian of the two linear functions.

In the example we cannot descend from the two quantics U, V to the single quantic U (for putting V=U the covariant vanishes); but this is merely accidental, as appears by considering a different lineo-linear covariant $\overline{12}^2 UV$, the developed expression of which is

$$\partial_x^2 U . \partial_y^2 V - 2\partial_x\partial_y U . \partial_x\partial_y V + \partial_y^2 U . \partial_x^2 V.$$

In the particular case of two quadrics $(a, b, c \between x, y)^2$, $(a', b', c' \between x, y)^2$, the lineo-linear covariant becomes the lineo-linear invariant

$$ac'-2bb'+ca'.$$

If we have V=U, then the lineo-linear covariant gives the quadricovariant

$$\partial_x^2 U . \partial_y^2 U - (\partial_x\partial_y U)^2$$

of the single quantic U (such quadricovariant is in fact the Hessian); and if in the last-mentioned formula we put for U the quadric $(a, b, c \between x, y)^2$, or what is the same thing, if in the expression of the lineo-linear invariant $ac'-2bb'+ca'$, we put the two quadrics equal to each other, we have the quadrinvariant

$$ac-b^2$$

of the single quadric.

70. The lineo-linear invariant $ab'-a'b$ of two linear functions may be considered as giving the lineo-linear covariant $\partial_x U . \partial_y V - \partial_y U . \partial_x V$ of the two quantics U and V, and in like manner the lineo-linear invariant $ac'-2bb'+ca'$ may be considered as giving the lineo-linear covariant $\partial_x^2 U . \partial_y^2 V - 2\partial_x\partial_y U . \partial_x\partial_y V + \partial_y^2 U . \partial_x^2 V$ of the quantics U, V. And generally, any invariant whatever of a quantic or quantics of a given order or orders leads to a covariant of a quantic or quantics of any higher order or orders: viz. the coefficients of the original quantic or quantics are to be replaced by the derived functions of the quantic or quantics of a higher order or orders.

71. The same thing may be seen by means of the theory of Emanants. In fact, consider any emanants whatever of a quantic or quantics; then, attending only to the facients of emanation, the emanants will constitute a system of quantics the coefficients of which are derived functions of the given quantic or quantics; the invariants of the system of emanants will be functions of the derived functions of the given quantic or quantics, and they will be covariants of such quantic or quantics; and we thus pass from the invariants of a quantic or quantics to the covariants of a quantic or quantics of a higher order or orders.

72. It may be observed also, that in the case where a tantipartite invariant, when the several quantics are put equal to each other, does not become equal to zero, we may pass back from the invariant of the single quantic to the tantipartite invariant of the system; thus the lineo-linear invariant $ac'-2bb'+ca'$ of two quadrics leads to the quadrinvariant $ac-b^2$ of a single quantic; and *conversely*, from the quadrinvariant $ac-b^2$ of a single quadric, we obtain by an obvious process of derivation the expression $ac'-2bb'+ca'$ of the lineo-linear invariant of two quadrics. This is in fact included in the more general theory explained, No. 64.

73. Reverting now to binary quantics, two quantics of the same order, even or odd, have a lineo-linear invariant. Thus the two quadrics

$$(a, b, c \between x, y)^2,\ (a', b', c' \between x, y)^2$$

have (it has been seen) the lineo-linear invariant

$$ac'-2bb'+ca';$$

and in like manner the two cubics

$$(a, b, c, d \between x, y)^3,\ (a', b', c', d' \between x, y)^3$$

have the lineo-linear invariant

$$ad'-3bc'+3cb'-da',$$

which examples are sufficient to show the law.

74. The lineo-linear invariant of two quantics of the same odd order is a combinant, but this is not the case with the lineo-linear invariant of two quantics of the same even order. Thus the last-mentioned invariant is reduced to zero by each of the operations

$$a\partial_{a'}+b\partial_{b'}+c\partial_{c'}+d\partial_{d'}$$

and

$$a'\partial_a+b'\partial_b+c'\partial_c+d'\partial_d;$$

but the invariant

$$ac'-2bb'+ca'$$

is by the operations

$$a\partial_{a'}+b\partial_{b'}+c\partial_{c'}$$

and

$$a'\partial_a+b'\partial_b+c'\partial_c$$

reduced respectively to

$$2(ac-b^2)$$

and

$$2(a'c'-b'^2).$$

75. For two quantics of the same odd order, when the quantics are put equal to each other, the lineo-linear invariant vanishes; but for two quantics of the same even order, when these are put equal to each other, we obtain the quadrinvariant of the single quantic. Thus the quadrinvariant of the quadric $(a, b, c \between x, y)^2$ is

$$ac-b^2;$$

and in like manner the quadrinvariant of the quartic $(a, b, c, d, e \between x, y)^4$ is

$$ae-4bd+3c^2.$$

76. When the two quantics are the first derived functions of the same quantic of any odd order, the lineo-linear invariant does not vanish, but it is not an invariant of the single quantic. Thus the lineo-linear invariant of

$$(a, b, c \between x, y)^2$$

and

$$(b, c, d \between x, y)^2$$

is

$$(ad-2bc+cb=)ad-bc,$$

which is not an invariant of the cubic

$$(a, b, c, d \between x, y)^3.$$

But for two quantics which are the first derived functions of the same quantic of any even order, the lineo-linear invariant is the quadrinvariant of the single quantic. Thus the lineo-linear invariant of

$$(a, b, c, d \between x, y)^3$$

and

$$(b, c, d, e \between x, y)^3$$

is

$$(ae-3bd+3c^2-db=)ae-4bd+3c^2,$$

which is the quadrinvariant of the quartic

$$(a, b, c, d, e \between x, y)^4.$$

77. I do not stop to consider the theory of the lineo-linear covariants of two quantics, but I derive the quadricovariants of a single quantic directly from the quadrinvariant. Imagine a quantic of any order even or odd. Its successive even emanants will be in regard to the facients of emanation quantics of an even order, and they will each of them have a quadrinvariant, which will be a quadricovariant of the given quantic. The emanants in question, beginning with the second emanant, are (in regard to the facients of the given quantic assumed to be of the order m) of the orders $m-2$, $m-4$, .. down to 1 or 0, according as m is odd or even, or writing successively $2p+1$ and $2p$ in the place of m, and taking the emanants in a reverse order, the emanants for a quantic of any odd order $2p+1$ are of the orders $1, 3, 5 \dots 2p-1$, and for a quantic of any even order $2p$, they are of the orders $0, 2, 4 .. 2p-2$. The quadricovariants of a quantic of an odd order $2p+1$, are consequently of the orders $2, 6, 10 \dots 4p-2$, and the quadricovariants of a quantic of an even order $2p$, are of the orders $0, 4, 8 \dots 4p-4$. We might in each case carry the series one step further, and consider a quadricovariant of the order $4p+2$, or (as the case may be) $4p$, which arises from the 0th emanant of the given quantic; such quadricovariant is, however, only the square of the given quantic.

78. In the case of a quantic of an evenly even order (but in no other case) we have a quadricovariant of the same order with the quantic itself. We may in this case form the lineo-linear invariant of the quantic and the quadricovariant of the same order: such lineo-linear invariant is an invariant of the given quantic, and it is of the degree 3 in the

coefficients, that is, it is a cubinvariant. This agrees with the before-mentioned theorem for the number of cubinvariants.

79. In the case of the quartic $(a, b, c, d, e \between x, y)^4$, the cubinvariant is, by the preceding mode of generation, obtained in the form

$$e(ac-b^2)-4d\tfrac{2}{4}(ad-bc)+6c\tfrac{1}{6}(ae-4bd+3c^2)-4b\tfrac{2}{4}(be-cd)+a(ce-d^2),$$

which is in fact equal to

$$3(ace-ad^2-b^2e+2bcd-c^3)\,;$$

and omitting the numerical factor 3, we have the cubinvariant of the quartic.

80. In the case of a quantic of any order even or odd, the quadrinvariants of the quadricovariants are quartinvariants of the quantic. But these quartinvariants are not all of them independent, and there is no obvious method grounded on the preceding mode of generation for obtaining the number of the independent (asyzygetic) quartinvariants, and thence the number of the irreducible quartinvariants of a quantic of a given order.

81. I take the opportunity of giving some additional developments in relation to the discriminant of a quantic

$$(a, b, \ldots b', a' \between x, y)^m.$$

To render the signification perfectly definite, it should be remarked that the discriminant contains the term $a^{m-1}\, a'^{m-1}$, and that the coefficient of this term may be taken to be $+1$. It was noticed in the introductory memoir, that, by JOACHIMSTHAL'S theorem, the discriminant, on putting $a=0$, becomes divisible by b^2, and that throwing out this factor it is to a numerical factor *près* the discriminant of the quantic of the order $(m-1)$ obtained by putting $a=0$ and throwing out the factor x; and it was also remarked, that this theorem, combined with the general property of invariants, afforded a convenient method for the calculation of the discriminant of a quantic when that of the order immediately preceding is known. Thus let it be proposed to find the discriminant of the cubic

$$(a, b, c, d \between x, y)^3.$$

Imagine the discriminant expanded in powers of the leading coefficient a in the form

$$\mathrm{A}a^2+\mathrm{B}a+\mathrm{C},$$

then this function *quà* invariant must be reduced to zero by the operation $3b\partial_a+2c\partial_b+d\partial_c$; or putting for shortness $\nabla=2c\partial_b+d\partial_c$, the operation is $\nabla+3b\partial_a$, and we have

$$\left.\begin{array}{l} a^2\nabla\mathrm{A}+a\nabla\mathrm{B}\ +\nabla\mathrm{C} \\ \qquad +a6b\mathrm{A}+3b\mathrm{B} \end{array}\right\}=0,$$

and consequently

$$\mathrm{B}=-\frac{1}{3b}\nabla\mathrm{C},\ \ \mathrm{A}=-\frac{1}{6b}\nabla\mathrm{B},\ \ \nabla\mathrm{A}=0.$$

But C is equal to b^2 into the discriminant of $(3b, 3c, d \between x, y)^2$, that is, its value is $b^2(12bd-9c^2)$, or throwing out the factor 3, we may write

$$\mathrm{C}=4b^3d-3b^2c^2\,;$$

this gives

$$B=-\frac{1}{3b}(-6b^2cd+24b^2cd-12bc^3),$$

or reducing,

$$B=-6bcd+4c^3;$$

and thence

$$A=-\frac{1}{6b}(-6bd^2+12c^2d-12c^2d),$$

or reducing, $$A=d^2,$$

which verifies the equation $\nabla A=0$, and the discriminant is, as we know,

$$a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2.$$

82. If we consider the quantic $(a, b..a'\between x, 1)^m$ as expressed in terms of the roots in the form $a(x-\alpha y)(x-\beta y)\ldots$, then the discriminant $(=a^{m-1}a'^{m-1}+$ &c. as above) is to a factor *près* equal to the product of the squares of the differences of the roots, and the factor may be determined as follows: viz. denoting by $\zeta(\alpha, \beta, \ldots)$ the product of the squares of the differences of the roots, we may write

$$a^{2m-2}\zeta(\alpha, \beta, ..)=\mathrm{N}(a^{m-1}a'^{m-1}+\&c.),$$

where N is a number; and then considering the equation $x^m-1=0$, we have to determine N the equation

$$\zeta(\alpha, \beta, ..)=(-)^{m-1}\mathrm{N}.$$

But in general

$$\zeta(\alpha, \beta..)=(-)^{\frac{1}{2}m(m-1)}(\alpha-\beta)(\alpha-\gamma)..(\beta-\alpha)(\beta-\gamma)\ldots$$

and if

$$\phi x=(x-\alpha)(x-\beta)..,$$

then

$$(\alpha-\beta)(\alpha-\gamma)..=\phi'\alpha, \&c.,$$

or

$$\zeta(\alpha, \beta, \ldots)=(-)^{\frac{1}{2}m(m-1)}\phi'\alpha\phi'\beta..;$$

here

$$\phi x=x^m-1,\ \phi' x=mx^{m-1},$$

and therefore

$$\phi'\alpha\phi'\beta..=m^m(\alpha\beta\gamma\ldots)^{m-1},$$

but

$$(-)^m\alpha\beta\gamma..=-1,$$

or

$$\alpha\beta\gamma..=(-)^{m-1}1,$$

and

$$\phi'\alpha\phi'\beta..=(-)^{(m-1)^2}m^m=(-)^{m-1}m^m;$$

whence

$$\zeta(\alpha, \beta\ldots)=(-)^{m-1+\frac{1}{2}m(m-1)}m^m=(-)^{m-1}\mathrm{N},$$

or

$$\mathrm{N}=(-)^{\frac{1}{2}m(m-1)}m^m,$$

and consequently

$$a^{m-2}\zeta(\alpha, \beta, ..)=(-)^{\frac{1}{2}m(m-1)}m^m(a^{m-1}a'^{m-1}+\&c.),$$

or what is the same thing, the value of the discriminant $\square(=a^{m-1}a'^{m-1}+\text{\&c.})$ is

$$(-)^{\frac{1}{2}m(m-1)}m^{-m}a^{m-2}\zeta(\alpha, \beta, \ldots).$$

It would have been allowable to define the discriminant so as that the leading term should be

$$(-)^{\frac{1}{2}m(m-1)}a^{m-1}a'^{m-1},$$

in which case the discriminant would have constantly the same sign as the product of the squared differences; but I have upon the whole thought it better to make the leading term of the discriminant always positive.

83. A quantic of an even order $2p$ has an invariant of peculiar simplicity, viz. the determinant the terms of which are the coefficients of the pth differential coefficients, or derived functions of the quantic with respect to the facients; such invariant may also be considered as a tantipartite invariant of the pth emanants. Thus the sextic

$$(a, b, c, d, e, f, g \between x, y)^6$$

has for one of its invariants, the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

The invariant in question is termed by Professor SYLVESTER the *Catalecticant*.

84. Professor SYLVESTER also remarked, that we may from the catalecticant form a function containing an indeterminate quantity λ, such that the coefficients of the different powers of λ are invariants of the quantic; thus for the sextic, the function in question is

$$\begin{vmatrix} a & , b & , c & , d-\lambda \\ b & , c & , d+\frac{1}{3}\lambda, & e \\ c & , d-\frac{1}{3}\lambda, & e & , f \\ d+\lambda, & e & , f & , g \end{vmatrix}$$

where the law of formation is manifest; the terms in the sinister diagonal are modified by annexing to their numerical submultiples of λ with the signs $+$ and $-$ alternately, and in which the multipliers are the reciprocals of the binomial coefficients. The function so obtained is termed the *Lambdaic*.

85. If we consider a quantic of an odd order, and form the catalecticant of the penultimate emanant, we have the covariant termed the Canonisant. Thus in the case of the quintic

$$(a, b, c, d, e, f \between x, y)^5,$$

the canonisant is

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix}$$

which is equivalent to

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix}$$

and a like transformation exists with respect to the canonisant of a quantic of any odd order whatever. The canonisant and the lambdaic (which includes of course the catalecticant) form the basis of Professor SYLVESTER'S theory of the *Canonical Forms* of quantics of an odd and an even order respectively.

86. There is another family of covariants which remains to be noticed. Consider any two quantics of the same order,

$$(a, b, ..\between x, y)^m,$$
$$(a', b', ..\between x, y)^m,$$

and join to these a quantic of the next inferior order,

$$(u, v, ..\between y, -x)^{m-1},$$

where the coefficients $(u, v, ..)$ are considered as indeterminate, and which may be spoken of as the adjoint quantic.

Take the odd lineo-linear covariants (viz. those which arise from the odd emanants) of the two quantics; the term arising from the $(2i+1)$th emanants is of the form

$$(\text{A}, \text{B}, ..\between x, y)^{2(m-1-2i)},$$

where (A, B, ..) are lineo-linear functions of the coefficients of the two quantics.

Take also the quadricovariants of the adjoint quantic; the term arising from the $(2i-m)$th emanant is of the form

$$(\text{U}, \text{V}, ..\between x, y)^{2(m-1-2i)},$$

where (U, V, ..) are quadric functions of the indeterminate coefficients $(u, v, ..)$. We may then form the quadrinvariant of the two quantics of the order $2(m-1-2i)$: this will be an invariant of the two quantics and the adjoint quantic, lineo-linear in the coefficients of the two quantics and of the degree 2 in regard to the coefficients $(u, v, ..)$ of the adjoint quantic; or treating the last-mentioned coefficients as facients, the result is a lineo-linear m-ary quadric of the form

$$(\mathfrak{A}, \mathfrak{B}, ..\between u, v, ..)^2,$$

viz. in this expression the coefficients $\mathfrak{A}, \mathfrak{B}, ..$ are lineo-linear functions of the coefficients of the two quantics. And giving to i the different admissible values, viz. from $i=0$ to $i=\frac{1}{2}m-1$ or $\frac{1}{2}(m-1)-1$, according as m is even or odd, the number of the functions obtained by the preceding process is $\frac{1}{2}m$ or $\frac{1}{2}(m-1)$, according as m is even or odd. The functions in question, the theory of which is altogether due to Professor SYLVESTER, are termed by him *Cobezoutiants*; we may therefore say that a cobezoutiant is an invariant of two quantics of the same order m, and of an adjoint quantic of the next preceding order $m-1$, viz. treating the coefficients of the adjoint quantic as the facients of the cobezoutiant, the cobezoutiant is an m-ary quadric, the coefficients of which are lineo-linear functions of the coefficients of the two quantics, and the number of the cobezoutiants is $\frac{1}{2}m$ or $\frac{1}{2}(m-1)$, according as m is even or odd.

87. If the two quantics are the differential coefficients, or first derived functions (with respect to the facients) of a single quantic

$$(a, b, \ldots \between x, y)^m,$$

then we have what are termed the *Cobezoutoids* of the single quantic, viz. the cobezoutoid is an invariant of the single quantic of the order m, and of an adjoint quantic of the order $(m-2)$; and treating the coefficients of the adjoint quantic as facients, the cobezoutoid is an $(m-1)$ary quadric, the coefficients of which are quadric functions of the coefficients of the given quantic. The number of the cobezoutoids is $\frac{1}{2}(m-1)$ or $\frac{1}{2}(m-2)$, according as m is odd or even.

88. Consider any two quantics of the same order,

$$(a, \ldots \between x, y)^m, \quad (a', \ldots \between x, y)^m,$$

and introducing the new facients (X, Y), form the quotient of determinants,

$$\begin{vmatrix} (a, \ldots \between x, y)^m, & (a', \ldots \between x, y)^m \\ (a, \ldots \between X, Y)^m, & (a', \ldots \between X, Y)^m \end{vmatrix} \div \begin{vmatrix} x, & y \\ X, & Y \end{vmatrix}$$

which is obviously an integral function of the order $(m-1)$ in each set of facients separately, and lineo-linear in the coefficients of the two quantics; for instance, if the two quantics are

$$(a, b, c, d \between x, y)^3,$$
$$(a', b', c', d' \between x, y)^3,$$

the quotient in question may be written

$$\left(\begin{array}{ccc} 3(ab'-a'b), & 3(ac'-a'c), & ad'-a'd \\ 3(ac'-a'c), & ad'-a'd+9(bc'-b'c), & 3(bd'-b'd) \\ ad'-a'd, & 3(bd'-b'd), & 3(cd'-c'd) \end{array}\right| \between x, y \overset{2}{\between} X, Y)^2$$

The function so obtained may be termed the *Bezoutic Emanant* of the two quantics.

89. The notion of such function was in fact suggested to me by BEZOUT'S abbreviated process of elimination, viz. the two quantics of the order m being put equal to zero, the process leads to $(m-1)$ equations each of the order $(m-1)$: these equations are nothing else than the equations obtained by equating to zero the coefficients of the different terms of the series $(X, Y)^{m-1}$ in the Bezoutic emanant, and the result of the elimination is consequently obtained by equating to zero the determinant formed with the matrix which enters into the expression of the Bezoutic emanant. In other words, this determinant is the Resultant of the two quantics. Thus the resultant of the last-mentioned two cubics is the determinant

$$\left|\begin{array}{ccc} 3(ab'-a'b), & 3(ac'-a'c), & ad'-a'd \\ 3(ac'-a'c), & ad'-a'd+9(bc'-b'c), & 3(bd'-b'd) \\ ad'-a'd, & 3(bd'-b'd), & 3(cd'-c'd) \end{array}\right|$$

90. If the two quantics are the differential coefficients or first derived functions (with respect to the facients) of a single quantic of the order m, then we have in like manner the *Bezoutoidal Emanant* of the single quantic; this is a function of the order $m-2$ in each set of facients, and the coefficients whereof are quadric functions of the coefficients of the single quantic. Thus the Bezoutoidal emanant of the quartic

$$(a, b, c, d, e\between x, y)^4$$

is

$$\left(\begin{array}{ccc} 3(ac-b^2), & 3(ad-bc), & ae-bd \\ 3(ad-bc), & ae+8bd-9c^2, & 3(be-cd) \\ ae-bd, & 3(be-cd), & 3(ce-d^2) \end{array}\right|\between x, y \overset{2}{\between} X, Y)^2$$

and of course the determinant formed with the matrix which enters into the expression of the Bezoutoidal Emanant, is the discriminant of the single quantic.

91. Professor SYLVESTER forms with the matrix of the Bezoutic emanant and a set of m facients $(u, v, \ldots)$ an m-ary quadric function, which he terms the *Bezoutiant*. Thus the Bezoutiant of the before-mentioned two cubics is

$$\left(\begin{array}{ccc} 3(ab'-a'b), & 3(ac'-a'c), & ad'-a'd \\ 3(ac'-a'c), & ad'-a'd+9(bc'-b'c), & 3(bd'-b'd) \\ ad'-a'd, & 3bd'-b'd, & 3(cd'-c'd) \end{array}\right|\between u, v, w)^2;$$

and in like manner with the Bezoutoidal emanant of the single quantic of the order m and a set of $(m-1)$ new facients $(u, v, \ldots)$, an $(m-1)$ary quadric function, which he terms the *Bezoutoid*. Thus the Bezoutoid of the before-mentioned quartic is

$$\left(\begin{array}{ccc} 3(ac-b^2), & 3(ad-bc), & ae-bd \\ 3(ad-bc), & ae+8bd-9c^2, & 3(be-cd) \\ ae-bd, & 3(be-cd), & 3(ce-d^2) \end{array}\right|\between u, v, w)^2;$$

And to him is due the important theorem, that the Bezoutiant is an invariant of the two quantics of the order m and of the adjoint quantic $(u, v, \ldots\between y, -x)^{m-1}$, being in fact a linear function with mere numerical coefficients, of the invariants called Cobezoutiants, and in like manner that the Bezoutoid is an invariant of the single quantic of the order m and of the adjoint quantic $(u, v, \ldots\between y, -x)^{m-2}$, being a linear function with mere numerical coefficients of the invariants called Cobezoutoids.

The modes of generation of a covariant are infinite in number, and it is to be anticipated that, as new theories arise, there will be frequent occasion to consider new processes of derivation, and to single out and to define and give names to new covariants. But I have now, I think, established the greater part by far of the definitions which are for the present necessary.